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MULTI-ERGODIC ASPECTS OF HAMILTONIAN DYNAMICS

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ABSTRACT

The origin of the f^{-p} power spectrum in hamiltonian systems is discussed from the viewpoint of the stagnant motions in the transition regime between chaos and torus. The essential point is that the purely stagnant motions are non-stationary ($p=2$), and as the result the global chaos in hamiltonian systems becomes multi-ergodic in general. The distribution of Lyapunov exponents and the fluctuation of the PSD function are discussed in relation to the large deviation theory.

1. Introduction

Chaotic motions in hamiltonian systems often reveal a long time tail, e.g., the f^{-p} power spectrum. A fundamental mechanism to create such a long time memory is the stagnant effect in the boundary layer between chaos and torus. The transition regime between chaos and torus is called the stagnant layer in this paper.

An essential feature of the stagnant layer is the perpetual stability of orbits, that was proved by Nekhoroshev for the first time [1]. Applying the Nekhoroshev's theorem to the transition layer between chaos and torus, a statistical universal law was derived in the previous paper [2]. Consider the stagnant layer coordinate r as is shown in Fig.1, where r stands for the phenomenological depth of the stagnant layer. A representative orbit C stays for a long time near the outermost KAM surface before it escapes from the stagnant layer. The first passage time T necessary for orbits to cross over a threshold Δ r obeys to a universal distribution P(T) given by,

$$P(T) \sim 1/(T\log T) \tag{1}$$

Numerical results are well in line with the theoretical estimation by eq.(1) [3].

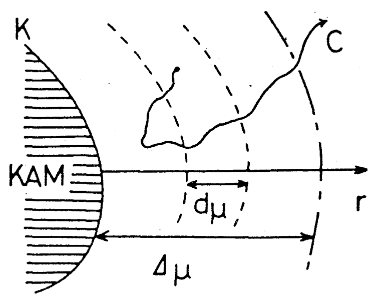

Fig. 1. Schematic picture of stagnant layer.

Universal features of a stagnant layer can be derived from the following scaling assumptions; first of all we assume that the invariant measure P(r) which describes stagnant motions is strongly localized near the final KAM,

$$P(r) \sim r^{-A} , \tag{2}$$

and that the Lyapunov exponent which characterizes the orbital unfolding in the tangent space satisfies,

$$\lambda \sim r^{d} , \tag{3}$$

where r stands for the distance in action space from the outermost KAM surface, A and d positive constants. Then the distribution of the Lyapunov exponent P(λ) becomes,

$$P(\lambda) \sim \lambda^{-D} , \tag{4}$$

with D=1-(A-1)/d. Furthermore, the distribution of the first passage time P(T) becomes,

$$P(T) \sim T^{-B} \quad , \tag{5}$$

with B=2-D . Here $T\lambda \sim 1$ is used. The last estimation is consistent with eq.(1) when D=1 holds.

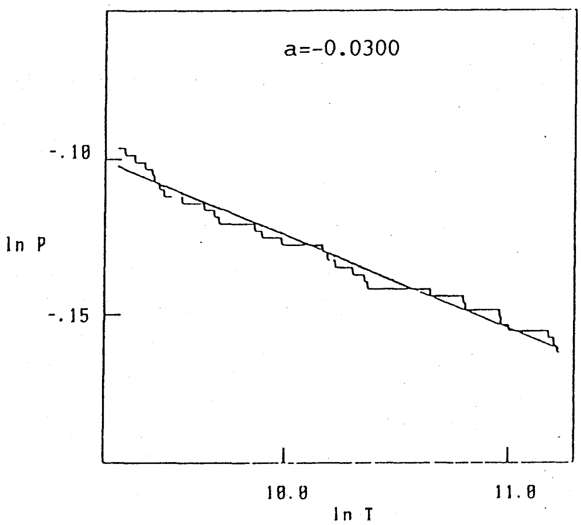

Fig.2 The first passage time distribution (a=1-B)

Figure 2 shows a numerical result of the first passage time distribution for the generalized billiard system [3]. Then the power spectrum S(f) of the variable Y = -log r reveals,

$$S(f) \sim f^{-2} \quad . \tag{6}$$

The essential feature of the stagnant motions are non-stationarity or non-recurrence, e.g., the mean first passage time is infinite and the probability measure of eq.(2) is not normalizable.

2. From Non-stationarity to Multi-ergodicity

Figure 3 shows the phase portrait of the standard mapping,

$$Y(n+2)=2Y(n+1)-Y(n)+K/2\pi \sin[Y(n+1)] \quad (\mathrm{mod.}1) \ , \tag{7}$$

with K=0.5. Torus regions are distributed in a quite fractal manner. Each torus is wrapped up with the inherent stagnant layer as was discussed before. A chaotic motion is trapped in every stagnant layer for long time. As the result the PSD function S(f) reveals many singularities at the resonant frequencies f_i's.

$$S(f) \sim \sum_i |f - f_i|^{-p_i} \tag{8}$$

Here f_i is the characteristic frequency in the i-th stagnant layer, and p_i stands for the singularity index. Figure 4 shows the mean power spectrum <S(f)> obtained numerically, where < > implies the average value over the asymptotic measure of an orbit considered. The fourier tranaformation is used as follows,

$$C(k,Y) = (1/\sqrt{N}) \sum_j Y(j)\exp[-i2\pi kj/N] \quad , \tag{9}$$

and the PSD function $S(f,Y)=|C(k,Y)|^2$ and the frequency f=k/N.

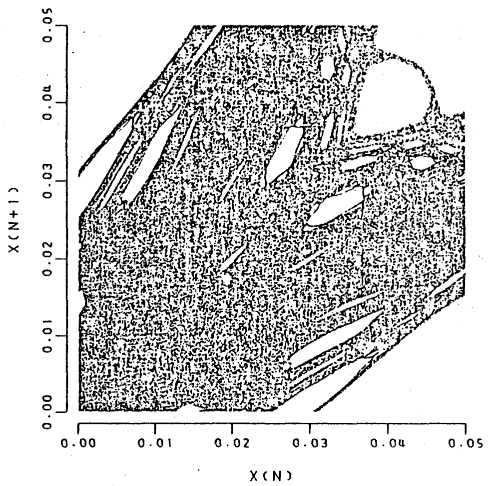

Fig.3 The phase portrait of the standard mapping(K=0.5).

When we consider the coarse-grained sampling process defined by $Y'(m)=Y(\ell m+j)$ $(j=1,2,\ldots;j=0,1,2,\ldots,\ell-1;\ \ell=\text{integer})$, the power spectrum or the fourier component C'(k',Y') is given by C(k,Y) as follows,

$$C(k,Y) = \sum_{j=0}^{\ell-1} \sum_{m=1}^{N/\ell} Y(\ell m+j)\exp[-i2\pi k(\ell m+j)/N], \tag{10}$$

$$= C'(k',Y')$$

with $k'=k-N/\ell$. Therefore, the singularity at $f=\ell/N$ in eq.(8) is rewritten into the singularity at $f'=k'\ell/N=0$. This enables us to consider that each singularity in eq.(8) is originated from the stagnant motion near the final KAM torus whose characteristic frequency is f_i (i=1,2,...). In other words, the multi-singularity in the PSD function implies the coexistence of many asymptotic measures which are strongly localized near the final KAM tori. The coexistence of such asymptotic measures is called the multi-ergodicity for short. From eq.(6) the spectral indeces should be equal $p_i=2$, if an orbit is perpetually trapped in the stagnant layer. But in the numerical calculations the values are fluctuating, and the statistical distribution must be considered, since the orbit can not be confined in the deep inside of each stagnant layer. The distribution function of the indeces P(p) is considered to be a characteristics of the multi-ergodic motion.

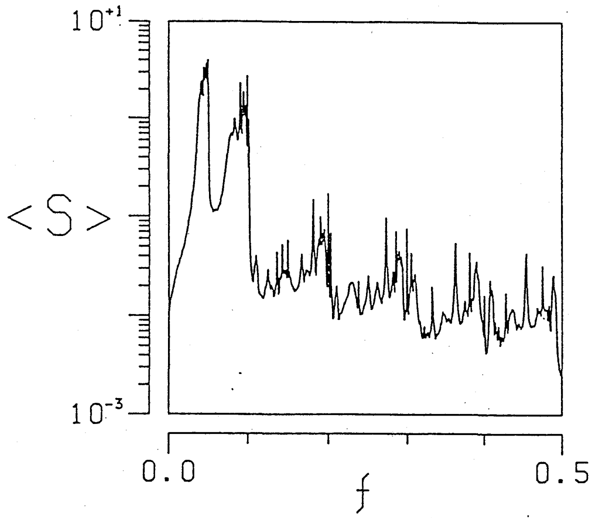

Fig.4 The mean power spectrum at K=1/2

Equation (8) persists that a lot of ordered cycles with different periods are immersed in a multi-ergodic orbit. This situattion is quite similar with the critical phenomena in statistical mechanics, where the coexistence of many phases can be explained even in the thermodynamical limit.

3. Large Deviation Properties of Multi-ergodic Motions

The anomalous fluctuations of multi-ergodic motions are characterized by the large deviation property of the PSD function [4]. Let us consider the following scaling form,

$$\langle S(f)\rangle \sim N^{-\mu(f)+1} \tag{11}$$

$\mu(f)$ is the scaling index of the PSD function $\langle S(f)\rangle$. For the ordinary random process, the index μ is unity in whole frequency domain $0<f<1$, but for the multi-ergodic motion the index is not uniform.

Figure 5 shows the distribution of the index $P(\mu)$ which will be scaled as $F(\mu) \sim -\log P(\mu)/\log N$. The dimension spectrum $F(\mu)$ was discussed in [3] from the viewpoint of multi-fractals,but it seems to be better to discuss it in the framework of the large deviation theory. The details were discussed in [3].

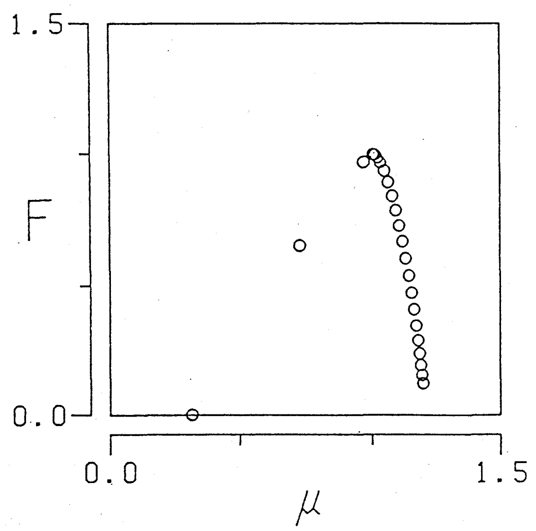

Fig.5 The dimension spectrum at K=1/2

The convergence of the Lyapunov exponent $\lambda(T)$ is also curious as the result of the multi-ergodicity.

$$\lambda(T) = (1/T)\log(|\xi(T)|/|\xi(0)|) \quad , \qquad (12)$$

where $\xi(T)$ is the tangent vector at the time T and the initial ensemble is the asymptotic measure of an orbit under consideration. Figure 6 is the distribution of the Lyapunov exponent $P(\lambda, T)$. At $T=10^5$ the distribution has three peaks, but only one dominant peak remains at $T=10^6$.

It is surmised from Fig.6(a) that the distribution is consisted of three factors $P_i(\lambda)$ (i=1,2,3),

$$P(\lambda) = aP_1(\lambda) + bP_2(\lambda) + cP_3(\lambda) \quad , \qquad (13)$$

where P_i is normalized in each domain and (a,b,c) are global normalization constants;

$$P_1(\lambda) \quad ; \quad 0 < \lambda < 0.005$$

$$P_2(\lambda) \quad ; \quad 0.005 < \lambda < 0.02$$

$$P_3(\lambda) \quad ; \quad 0.02 < \lambda < 0.1$$

We can surmise that three components mentioned above have defferent large deviation properties. Indeed, the phase points which belong to each distribution P_i (i=1,2,3) are clearly separated in phase space as is shown in Fig.7 .

The lowest peak distribution P_1 is originated from the stagnant motions which strongly localized around KAM tori (Fig.7-a), and it obeys to a hyperbolic law of eq.(4) with D=0.35 at $T=10^5$. The last peak distribution P_3 is coming from the chaotic motions far from KAM tori (Fig.7(b), where the orbits are extended in a wide chaotic region. The second peak distribution P_2 is the most ambiguous one. It is difficult to decide its origin in phase space definitely. The conclusion has not yet been obtained, but it is surmised that P_2 characterizes the diffusion process across the cantori or the narrow gates in phase space, which can confine the orbit in a small sub-space for a long but finite periods.

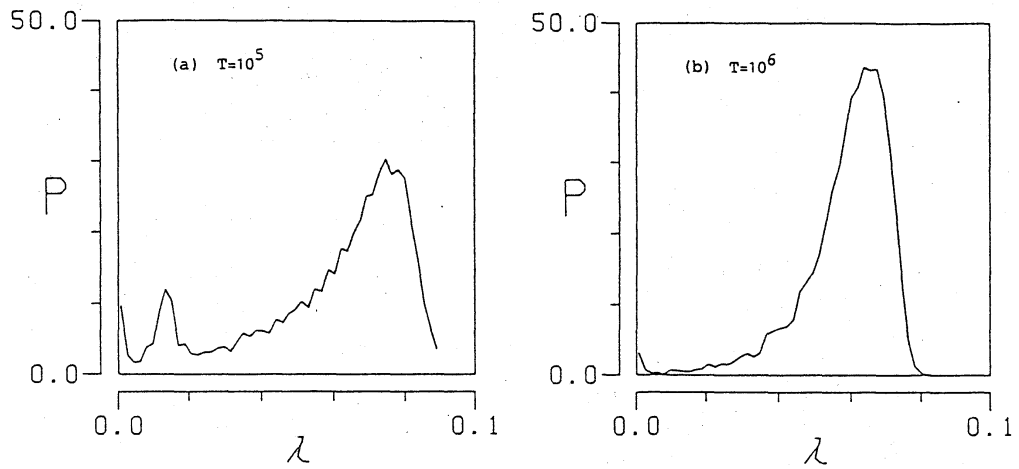

Fig.6 The distribution of the Lyapunov exponent at K=1/2

The large deviation of each contribution from P_i must be elucidated in terms of the entropy $H(\lambda)$ defined by,

$$P_i(\lambda) \sim \exp[-T^{q_i} H_i(\lambda)] \quad , \tag{14}$$

where q stands for the convergence speed for each contribution. In the thermodynamical limit of $T \to \infty$, the distribution $P(\lambda)$ approaches to a single component with the minimum value of q_i's . In the present simulation, P_3 will be realized when T goes to infinity, and then the universal form of the entropy is derived numerically as well as theoretically,

$$H(X) = 1 - X + \exp[X] \quad , \tag{15}$$

under an appropriate scale transformation, and that the normal exponential convergence obtained , i.e., q_3=1. The details will be discussed precisely in the forthcoming paper [6].

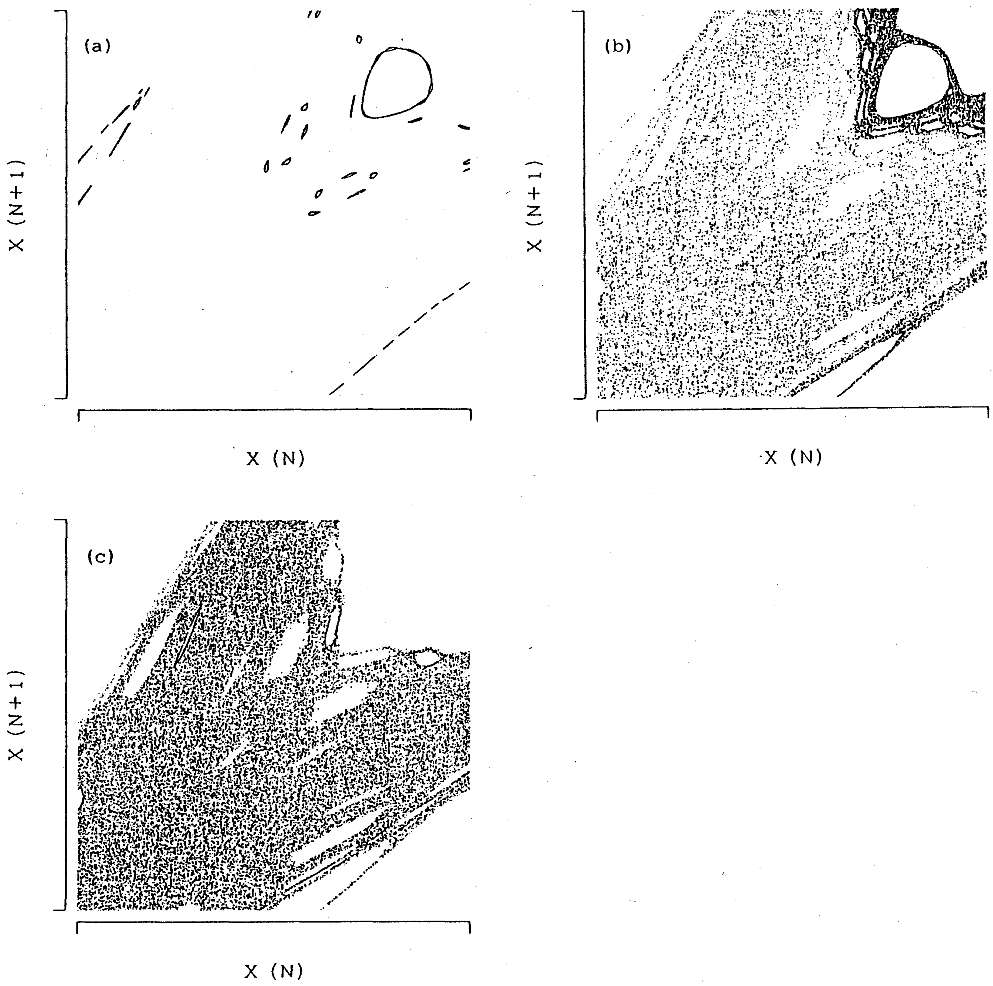

Fig.7 The orbital distributions which contribute to the three dominant components in Fig.6. (a):$P_1(\lambda)$,(b):$P_2(\lambda)$,(c):$P_3(\lambda)$.

The origin of the long time tails in hamiltonian systems was discussed by Geisel [5] along the same line as is explained in this article. But the main purpose of our study is to research the origin of non-stationary fluctuations beyond the infra-red crisis limit of $\langle S(f)\rangle \sim f^{-1}$.

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